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Photon Splitting in a Strong Magnetic Field: Recalculation and Comparison With Previous Calculations

Stephen L. Adler
Institute for Advanced Study
Princeton, NJ 08540

Christian Schubert
Humboldt Universitität zu Berlin
Invalidenstr. 110, D-10115 Berlin, Germany

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Abstract

We recalculate the amplitude for photon splitting in a strong magnetic field below the pair production threshold, using the worldline path integral variant of the Bern–Kosower formalism. Numerical comparison (using programs that we have made available for public access on the Internet) shows that the results of the recalculation are identical to the earlier calculations of Adler and later of Stoneham, and to the recent recalculation by Baier, Milstein, and Shaisultanov.

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Photon splitting in a strong magnetic field is an interesting process, both from a theoretical viewpoint because of the relatively sophisticated methods needed to do the calculation, and because of its potential astrophysical applications. The first calculation to exactly include the corrections arising from nonzero photon frequency ω was given by Adler [1], who obtained the amplitude as a triple integral that is strongly convergent below the pair production threshold at $\omega = 2m$, and who included a numerical evaluation for the special case $\omega = m$. Subsequently, the calculation was repeated by Stoneham [2] using a different method, leading to a different expression as a triple integral, that has never been compared to the formula of Ref. [1] either analytically or numerically. Recently, a new calculation has been published by Mentzel, Berg, and Wunner [3] in the form of a triple infinite sum, and numerical evaluation of their formula by Wunner, Sang, and Berg [4] claims photon splitting rates roughly four orders of magnitude larger than those found in Ref. [1]. Since this result, if correct, would have important astrophysical implications, a recalculation by an independent method seems in order. We report the results of such a recalculation here, together with a numerical comparison of the resulting amplitude with those of Adler and of Stoneham, as well as with a recent recalculation independently carried out by Baier, Milstein, and Shaisultanov [5]. The comparison shows that these four independent calculations give precisely the same amplitude, showing no evidence of the dramatic energy dependent effects claimed in Refs. [3] and [4].

Our recalculation of the photon splitting amplitude uses a variant of the worldline path integral approach to the Bern–Kosower formalism [6, 7, 8, 9]. As is well known, the one loop QED effective action induced for the photon field by a spinor loop can be represented by the following double path integral,

$$
\begin{aligned}
\Gamma[A] = & -2\int_0^\infty \frac{ds}{s} e^{-m^2 s}\int \mathcal{D}x\mathcal{D}\psi \\
& \times\exp\Big[-\int_0^s d\tau\Big(\frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi} + ieA_\mu\dot{x}^\mu - ie\psi^\mu F_{\mu\nu}\psi^\nu\Big)\Big].
\end{aligned} \tag{1}
$$

Here s is the usual Schwinger proper–time parameter, the $x^\mu(\tau)$'s are the periodic functions from the circle with circumference s into spacetime, and the $\psi^\mu(\tau)$'s are antiperiodic and Grassmann–valued.

Photon scattering amplitudes are obtained by specializing the background to a sum of plane waves with definite polarizations. Both path integrals are then evaluated by one–dimensional perturbation theory, i.e. one obtains an integral representation for the N–photon amplitude by Wick–contracting N "photon vertex operators"

$$V = \int_0^T d\tau\, [\dot{x}^\mu \varepsilon_\mu - 2\mathrm{i}\psi^\mu\psi^\nu k_\mu\varepsilon_\nu]\exp[ikx(\tau)]. \tag{2}$$

The appropriate one–dimensional propagators are

$$\begin{aligned}\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle = -g^{\mu\nu}G_B(\tau_1,\tau_2) = &\quad -g^{\mu\nu}\Bigl[\mid \tau_1-\tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{s}\Bigr],\\ \langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle = \frac{1}{2}\,g^{\mu\nu}G_F(\tau_1,\tau_2) = &\quad \frac{1}{2}\,g^{\mu\nu}\mathrm{sign}(\tau_1-\tau_2)\,.\end{aligned} \tag{3}$$

The bosonic Wick contraction is actually carried out in the relative coordinate $y(\tau) = x(\tau) - x_0$ of the closed loop, while the (ordinary) integration over the average position $x_0 = \frac{1}{s}\int_0^s d\tau x(\tau)$ yields energy–momentum conservation.

To take the additional constant magnetic background field B into account, one chooses Fock–Schwinger gauge, where its contribution to the worldline Lagrangian becomes

$$\Delta\mathcal{L} = \frac{1}{2}iey^\mu F_{\mu\nu}\dot{y}^\nu - ie\psi^\mu F_{\mu\nu}\psi^\nu \quad . \tag{4}$$

Being bilinear, those terms can be simply absorbed into the kinetic part of the Lagrangian [9 ,10]. This leads to generalized worldline propagators defined by

$$\frac{1}{2}\Bigl(\frac{\partial^2}{\partial\tau^2} - 2ieF\frac{\partial}{\partial\tau}\Bigr)\mathcal{G}_B(\tau_1,\tau_2) = \delta(\tau_1-\tau_2) - \frac{1}{s}, \tag{5}$$

$$\frac{1}{2}\Bigl(\frac{\partial}{\partial\tau} - 2ieF\Bigr)\mathcal{G}_F(\tau_1,\tau_2) = \delta(\tau_1-\tau_2). \tag{6}$$

The solutions to these equations can be written in the form [11]

$$\mathcal{G}_B(\tau_1,\tau_2) = \frac{1}{2(eF)^2}\Bigl(\frac{eF}{\sin(esF)}\mathrm{e}^{-iesF\dot{G}_{B12}} + ieF\dot{G}_{B12} - \frac{1}{s}\Bigr), \tag{7}$$

$$\mathcal{G}_F(\tau_1,\tau_2) = G_{F12}\frac{\mathrm{e}^{-iesF\dot{G}_{B12}}}{\cos(esF)}\,, \tag{8}$$

(we have abbreviated $G_{Bij} := G_B(\tau_i, \tau_j)$, and a dot always denotes a derivative with respect to the first variable). Those expressions should be understood as power series in the field strength matrix. To obtain the photon splitting amplitude, we will use them for the Wick contraction of three vertex operators V_0 and $V_{1,2}$, representing the incoming and the two outgoing photons.

The calculation is greatly simplified by the special kinematics of this process. Energy–momentum conservation $k_0 + k_1 + k_2 = 0$ forces collinearity of all three four–momenta, so that, writing $-k_0 \equiv k \equiv \omega n$,

$$k_1 = \frac{\omega_1}{\omega} k, k_2 = \frac{\omega_2}{\omega} k; k^2 = k_1^2 = k_2^2 = k \cdot k_1 = k \cdot k_2 = k_1 \cdot k_2 = 0. \tag{9}$$

Moreover, a simple CP–invariance argument together with an analysis of dispersive effects [1] shows that there is only one allowed polarization case. This is the one where the incoming photon is polarized parallel to the plane containing the external field and the direction of propagation, and both outgoing ones are polarized perpendicular to this plane. This choice of polarizations leads to the further vanishing relations

$$\varepsilon_{1,2} \cdot \varepsilon_0 = \varepsilon_{1,2} \cdot k = \varepsilon_{1,2} \cdot F = 0 \quad . \tag{10}$$

In particular, we cannot Lorentz contract ε_1 with anything but ε_2. This leaves us with only a small number of nonvanishing Wick contractions:

$$\begin{aligned} \langle V_0 V_1 V_2 \rangle = i \exp\Bigl[\frac{1}{2} \sum_{i,j=0}^{2} \bar{\omega}_i \bar{\omega}_j n \mathcal{G}_{Bij} n\Bigr] \Biggl\{ \Bigl[\varepsilon_1 \ddot{\mathcal{G}}_{B12} \varepsilon_2 + \varepsilon_1 \mathcal{G}_{F12} \varepsilon_2 \bar{\omega}_1 \bar{\omega}_2 n \mathcal{G}_{F12} n\Bigr] \\ \times \Bigl[-\sum_{i=0}^{2} \bar{\omega}_i \varepsilon_0 \dot{\mathcal{G}}_{B0i} n + \bar{\omega}_0 \varepsilon_0 \mathcal{G}_{F00} n\Bigr] - \bar{\omega}_0 \bar{\omega}_1 \bar{\omega}_2 \varepsilon_1 \mathcal{G}_{F12} \varepsilon_2 \Bigl[n \mathcal{G}_{F10} \varepsilon_0 \, n \mathcal{G}_{F20} n - (1 \leftrightarrow 2)\Bigr] \Biggr\}. \end{aligned} \tag{11}$$

For compact notation we have defined $\bar{\omega}_0 = \omega, \bar{\omega}_{1,2} = -\omega_{1,2}$. This result has still to be multiplied by an overall factor of $\frac{(esB)\cosh(esB)}{(4\pi s)^2 \sinh(esB)}$, which by itself would just produce the Euler–Heisenberg Lagrangian, and here appears as the product of the two free Gaussian path integrals [8].

It is then a matter of simple algebra to obtain the following representation for the matrix element $C_2[\omega, \omega_1, \omega_2, B]$ appearing in Eq. (25) of [1]:

$$
\begin{aligned}
&C_2[\omega,\omega_1,\omega_2,B] = \frac{m^8}{4\omega\omega_1\omega_2}\int_0^\infty dss\frac{\mathrm{e}^{-m^2s}}{(esB)^2\sinh(esB)}\int_0^s d\tau_1\int_0^s d\tau_2\\
&\times\exp\Bigl\{-\frac{1}{2}\sum_{i,j=0}^{2}\bar\omega_i\bar\omega_j\Bigl[G_{Bij}+\frac{1}{2eB}\frac{\cosh(esB\dot G_{Bij})}{\sinh(esB)}\Bigr]\Bigr\}\\
&\times\biggl\{\Bigl[-\cosh(esB)\ddot G_{B12}+\omega_1\omega_2\Bigl(\cosh(esB)-\cosh(esB\dot G_{B12})\Bigr)\Bigr]\\
&\times\Bigl[\omega\Bigl(\coth(esB)-\tanh(esB)\Bigr)-\omega_1\frac{\cosh(esB\dot G_{B01})}{\sinh(esB)}-\omega_2\frac{\cosh(esB\dot G_{B02})}{\sinh(esB)}\Bigr]\\
&+\omega\omega_1\omega_2\frac{G_{F12}}{\cosh(esB)}\Bigl[\sinh(esB\dot G_{B01})\Bigl(\cosh(esB)-\cosh(esB\dot G_{B02})\Bigr)-\Bigl(1\leftrightarrow 2\Bigr)\Bigr]\biggr\}.
\end{aligned}
\tag{12}
$$

Here translation invariance in τ has been used to set the position τ_0 of the incoming photon equal to s. Coincidence limits have to be treated according to the rules $\dot G_B(\tau,\tau)=0, \dot G_B^2(\tau,\tau)=1$.

Alternatively, one may remove $\ddot G_{B12}$ by partial integration on the circle. This leads to the equivalent formula

$$
\begin{aligned}
&C_2[\omega,\omega_1,\omega_2,B] = \frac{m^8}{4}\int_0^\infty dss\,\mathrm{e}^{-m^2s}\frac{\cosh(esB)}{(esB)^2\sinh(esB)}\int_0^s d\tau_1\int_0^s d\tau_2\\
&\times\exp\Bigl\{-\frac{1}{2}\sum_{i,j=0}^{2}\bar\omega_i\bar\omega_j\Bigl[G_{Bij}+\frac{1}{2eB}\frac{\cosh(esB\dot G_{Bij})}{\sinh(esB)}\Bigr]\Bigr\}\\
&\times\biggl\{\Bigl[\dot G_{B12}\Bigl(\dot G_{B12}-\frac{\sinh(esB\dot G_{B12})}{\sinh(esB)}\Bigr)-\Bigl(1-\frac{\cosh(esB\dot G_{B12})}{\cosh(esB)}\Bigr)\Bigr]\\
&\times\Bigl[-\coth(esB)+\tanh(esB)+\frac{\omega_1}{\omega}\frac{\cosh(esB\dot G_{B01})}{\sinh(esB)}+\frac{\omega_2}{\omega}\frac{\cosh(esB\dot G_{B02})}{\sinh(esB)}\Bigr]\\
&+\dot G_{B12}\Bigl[\Bigl(\frac{\cosh(esB\dot G_{B02})}{\sinh(esB)}-\frac{1}{esB}\Bigr)\Bigl(\dot G_{B01}-\frac{\sinh(esB\dot G_{B01})}{\sinh(esB)}\Bigr)-\Bigl(1\leftrightarrow 2\Bigr)\Bigr]\\
&+\frac{1}{2}\dot G_{B12}\Bigl[\frac{\omega}{\omega_2}\Bigl(\dot G_{B01}-\frac{\sinh(esB\dot G_{B01})}{\sinh(esB)}\Bigr)-\Bigl(1\leftrightarrow 2\Bigr)\Bigr]\Bigl(-\coth(esB)+\frac{1}{esB}+\tanh(esB)\Bigr)\\
&+G_{F12}\Bigl[\frac{\sinh(esB\dot G_{B01})}{\cosh(esB)}\Bigl(1-\frac{\cosh(esB\dot G_{B02})}{\cosh(esB)}\Bigr)-\Bigl(1\leftrightarrow 2\Bigr)\Bigr]\biggr\}.
\end{aligned}
\tag{13}
$$

This form of the amplitude is less compact, but the integrand (apart from the exponential) is homogeneous in the ω_i.

Finally, let us remark that the analogous expression for scalar QED would be obtained by deleting all terms in Eq. (11) containing a $\mathcal{G}_F$, as well as the $\cosh(esB)$ appearing in the overall factor and the global factor of -2 in Eq. (1).

In order to compare the amplitudes of Eqs. (12) and (13) to those of Refs. [1], [2], and [5], we observe that both Eq. (12) and Eq. (13) can be written in the form

$$C_2[\omega,\omega_1,\omega_2,B] = \frac{m^8}{4B^2\omega\omega_1\omega_2}\int_0^\infty \frac{ds}{s} e^{-m^2 s} J_2(s,\omega,\omega_1,\omega_2,B) \ , \tag{14}$$

in which J_2 is independent of the electron mass m. Inspection shows that the amplitude expressions of Adler [1] and Baier, Milstein, and Shaisultanov [5] are already in the form of Eq. (14), while that of Stoneham [2] can be put in this form by doing an integration by parts in the proper time parameter s, using the identity

$$m^2 e^{-m^2 s} = -\frac{d}{ds} e^{-m^2 s} \tag{15}$$

to eliminate a term proportional to m^2 in the amplitude. In rewriting Stoneham's formulas in this form, we note that his $M_1(B)$ is what we are calling $C_2[\omega,\omega_1,\omega_2,B]$, and that there is an error of an overall minus sign in either his Eq. (37) or the first line of his Eq. (40). Similarly, in rewriting the formulas of Baier, Milstein, and Shaisultanov in this form, we note that their amplitude T is related to C_2 by

$$C_2[\omega,\omega_1,\omega_2,B] = \frac{\pi^{\frac{1}{2}} m^8}{4\alpha^3 B^3 \omega\omega_1\omega_2} T \quad . \tag{16}$$

Once all amplitudes are put in the form of Eq. (14), we can compare them by comparing the proper time integrand $J_2(s,\omega,\omega_1,\omega_2,B)$, which in each case involves only a double integral over a bounded domain. The only remaining subtlety is that we must remember that J_2 vanishes as $\omega\omega_1\omega_2$ for small photon energy; this is manifest in Eq. (13) above, but in Eq. (12) and the corresponding equations obtained from Refs. [1], [2], and [5], there is an apparent linear term in the frequencies which vanishes when the double integral is done exactly. In order to get robust results for small photon frequency when the double integral is done numerically, this linear term must first be subtracted away, by replacing expressions of the form

$$\int\int e^Q (L+C) \tag{17a}$$

with L, Q, and C respectively linear, quadratic, and cubic in the photon frequencies, by the subtracted expression

$$\int\int[(e^Q-1)L+C] \quad . \tag{17b}$$

This subtraction is already present in the expression of Eq. (25) of Ref. [1], and is discussed in the form of Eqs. (17a, b) in Ref. [5], and it also must be applied to Eqs. (37) and (39) of Ref. [2] after the integration by parts of Eq. (15) has been carried out. While in principle this subtraction should be applied to Eq. (12) above, it turns out not to be needed there, because the linear term in the frequencies involves only integrals of the general form

$$\int_0^s d\tau_1 f(s,\tau_1)\int_0^s d\tau_2[\delta(\tau_1-\tau_2)-1/s] \quad , \tag{18}$$

which is exactly zero using a discrete trapezoidal integration method when the δ function is discretized as a Kronecker delta. Thus Eq. (12) is robust for small photon frequencies as it stands, when used in conjunction with trapezoidal integration.

With these preliminaries out of the way, it is then completely straightforward to program the functions $J_2(s,\omega,\omega_1,\omega_2,B)$ for the five cases represented by the formulas of Adler [2], Stoneham [3], Eq. (12) of this paper, Eq. (13) of this paper, and Baier, Milstein, and Shaisultanov [5], with the result that they are all seen to be precisely the same; the residual errors approach zero quadratically as the integration mesh spacing approaches zero, as expected for trapezoidal integration. We have not carried out the s and ω_1 integrals needed to get the photon splitting absorption coefficient, since this was done in Ref. [1], with results confirmed by the more extensive numerical analysis given in Ref. [5]. However, anyone wishing to do this further computation can obtain our programs for calculating the proper time integrand J_2 by accessing S. L. A.'s home page on the Institute for Advanced Study web site (http://www.sns.ias.edu/∼adler/Html/photonsplit.html).

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